Perceptual Categories in Geometric Analogies

Mark A. Foltz
MIT Artificial Intelligence Laboratory
200 Technology Square
Cambridge, MA USA 02139
mfoltz@ai.mit.edu

Abstract

This work studies how the arrangements of shapes participate in geometric analogies. Perceptually significant categories of shape arrangements can be related in a lattice of their non-accidental (unlikely to arise by chance) features. In these lattices, the *codimension-change constraint* restricts the possible analogical matchings by requiring that they preserve the change in codimension, or the direction of movement, in the lattices.

It extends the work of Evans [Evans, 1968] on visual analogies, whose program considered similarity transformations to solve geometric analogy problems. Several arrangement-based examples illustrating the constraint are presented.

Introduction

A familiar style of question posed on intelligence tests ask the test-taker to form an analogy between two sets of geometric figures. The question is typically posed in a form like that found in Figure 1.

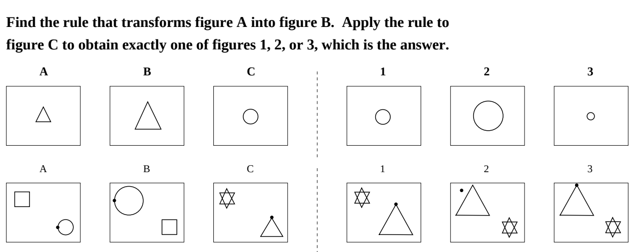

Figure 1: Typical geometric-analogy problems. The answers are figure 2 (top) and figure 3 (bottom).

This work analyzes geometric analogies as a problem-solving domain. Previous work has primarily considered the role of similarity transformations of shapes [Evans, 1968], but similarity transformations alone cannot specify the solution to the second problem in Figure 1, because the dot must remain on the vertex of the triangle. An additional constraint, the *codimension-change constraint*, is needed to require that the feature *dot-on-vertex* remain in both the base and target of the analogy.

This paper begins with a discussion of geometric analogies, then presents a problem-solving procedure which introduces the role shape arrangements play. The constraints between perceptual categories of arrangements that indicate correct analogies will be presented, and illustrated on a small set of geometric analogy problems.

Geometric analogies

This work focuses on analogies based only on the geometric properties of the figures, not on what the figures might represent. Geometric properties of a figure include the spatial arrangement of its component shapes, and the number of sides, location, size, and orientation of each of those shapes.

For this purpose, a figure is an arrangement of a small number of shapes in the plane, where a shape is constructed from a connected series of line segments. (For a more formal definition of *shape*, see Stiny [Stiny, 1980]). A point (or dot) is also considered a shape. Potential ambiguities in the decomposition of a figure of overlapping shapes will be avoided by selecting shapes from a fixed set in which these ambiguities do not arise. Also, the color and texture of shapes will not be considered as relevant to the analogies that might be constructed.

A geometric analogy problem is a series of pairs of figures, divided into a base pair and a set of target pairs. The base pair, labeled A and B, demonstrate the transformations applied to A that obtain B. The third figure, C, and each of the answer figures, 1, 2, . . ., N, are possible target pairs for the analogy. The target pairs are possible, analogous applications of the transformations to figure C. The task for the solver is to find the transformations, like "make the triangle larger" in the first problem of Figure 1, that describe the relationships between the shapes in figures A and B, substitute the shapes in figure C into those transformations, and select the answer figure that results.

This way of thinking about the problem, as using relationships in base pair of figures to select a target pair of figures, refers geometric analogy problems to analogies in the larger sense. A broad definition of analogy is the use of the relationships in one model to reason about another model. The models may be mental models [Johnson-Laird, 1983], conceptual graphs [Leishman, 1989], extensible-relations representations [Winston, 1980], or some other formal, visual, or natural-language representation. The base model is generalized, or a correspondence made between parts of the base model and parts of the target model, so that new inferences can be made about the parts of the target model.

Both approaches (*analogy-as-generalization* and *analogy-as-correspondence*) are represented in the literature on analogy in cognitive science and artificial intelligence [Falkenhainer et al., 1989, Winston, 1980]. For the purposes of this work, stating analogies as correspondences between the shapes in one figure and the shapes in another is more convenient. However, a restatement of those analogies as generalizations of transformations is easily possible.

Solving Geometric Analogies

To see how analogies are used in geometric analogy problems, I propose a procedure for solving these problems. The procedure finds three kinds of relationships among the shapes in the problem:

- *non-analogy correspondences*, which match instances of the same shape in the figures depicting transformations (A and B, C and 1, …, C and N);
- *analogy correspondences*, which match shapes transformed analogously in the problem (in figures A and C, B and 1, …, B and N).
- and *non-accidental features* in the arrangements of shapes in the figures.

The four steps of the procedure are:

1. Find non-analogy correspondences between the shapes in figures A and those in figure B, and between those in C and those in each of 1 through N.
2. For each non-analogy correspondence, find a shape transformation that indicates how the source shape is changed into its corresponding target shape. These transformations indicate how a shape in one figure relates to a shape in the other figure.
3. Find analogy-correspondences between the shapes in figures A and C, and between the shapes in B and those in each of 1 through N. These correspondences match pairs of analogous transformations in figures A and B, and figure C and n ($n \in 1 \ldots \text{N}$).
4. Account for any non-accidental features in the spatial arrangements of shapes in the problem.

The first three steps closely follow those in Evans' program [Evans, 1968]. The additional step accounts for the importance of the arrangement of shapes in each figures, and is intended to identify the correct solution in the second problem of Figure 1.

Note that a commitment is not being made to the order of execution of the four steps. Instead, these are four mutually interdependent subproblems that are solved, implicitly or explicitly, when the solver selects an answer figure.

The example problem worked throughout this section is illustrated in Table 1.

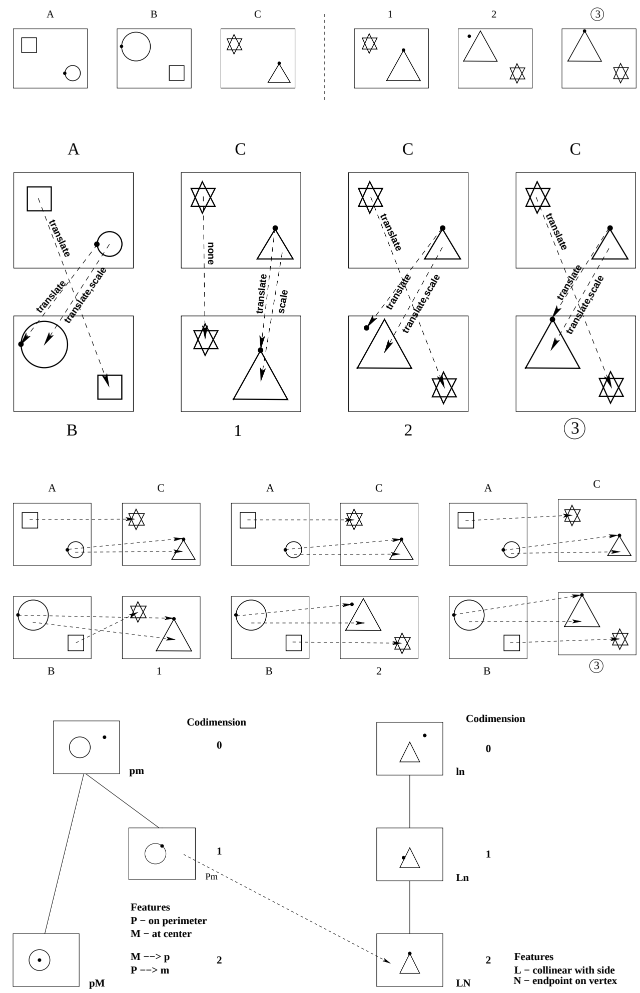

Table 1: Worked-through example. From top to bottom: Entire problem with answer circled; Steps 1, 2: Non-analogous correspondences with transformations; Step 3: Analogous correspondences; Step 4: Category lattices over relevant shapes, and the category correspondence that solves the problem.

Step 1: Non-analogous shape correspondences

In the first step, correspondences are found between shapes in figures A and B, and separately for the pairs of figures $\{C, 1\}$ through $\{C, N\}$. These correspondences associate each source shape in one figure with a target shape in the other figure.

These correspondences should give the simplest explanation of how the target shape relates to its source. For example, when choosing a target for a circle among the possible sources of an identical circle, a triangle, or a larger circle, the simplest explanation would match the circle to its identical twin. Each of these correspondences attempts to describe the simplest transformation that derives the target shape from its source.

Step 2: Similarity Transformations

For any shape, there are transformations that change the shape into other shapes. Although the full set of possible transformations is particular to the shape in question, the similarity transformations – translation, rotation, and scaling – apply to almost any shape. Transformations can deform shapes in other ways, for example by adding or removing sides from a polygon, or even adding or removing entire shapes from the figure (by correspondences from or to an "empty shape"). However, the similarity transformations provide many possibilities for geometric analogies, since they can be thought of as possible views of a rigid, planar object kept perpendicular to the observer.

Step 3: Analogous shape-correspondences

These correspondences capture the analogy by matching transformations found in steps 1 and 2 across the base and target pairs of figures. The criterion for forming these correspondences is, *match pairs of shapes in figures A and C, and in figures B and 1 through N, that have the most similar shape transformations between them.* The answer figure whose analogy-correspondences account for all of the shape transformations in the base pair of figures A and B is the best choice by this measure.

For the problem in question, analogy-correspondences are shown in the third row in the figure. For answer 1, there is no way to match the transformations between the shapes in figures A and B, $\{translate, translate, translate + scale\}$ to those between the shapes figures C and 1, $\{none, translate, scale\}$. For figures 2 and 3, however, there are one-to-one correspondences between all the transformations, leaving the solution ambiguous.

Steps 1-3 do not account for the placement of the dot on the perimeter of the circle in figures A and B, and its placement on the perimeter of the triangle in figures C and 3. This arrangement is maintained in the answer figure, 3, but not in figure 2, and so determines the correct answer. So, although steps 1-3 solve analogies based on transformations among individual shapes, the relationships among the shapes inside the figure indicate the solution to this example.

Step 4: Shape arrangements

What is needed is a way to relate the possible arrangements of shapes within a figure. Although similarity transformations can relate one shape to another through translation, scaling, and rotation, another representation is necessary for describing the perceptually significant categories of arrangements of multiple shapes.

Category Lattices and Analogies

Given two examples of the same shapes arranged in different ways, how would one tell if they were differed in a meaningful way, or not? In other words, how could one describe a shape arrangement as the placement of its shapes varies?

Some significant differences can occur from imprecise moves. For example, flipping a square and triangle arranged left-to-right so that they are arranged right-to-left does not depend crucially on the exact amount of translation that each shape undergoes; many possible translations suffice to distinguish the classes *left-to-right* and *right-to-left*.

Other arrangements, though, are *non-transverse*; a small perturbation in the position of one shape will change how the solver categorizes the arrangement. For example, moving a line whose endpoint is just at the center of a circle off of that point will lead the solver to classify the new arrangement with other arrangements in which the line is off-center.

Feldman shows that these non-transverse shape arrangements built of non-accidental (unlikely to arise by chance) features correspond to perceptual categories in observers [Feldman, 1992]. So, if there is a move from one such category to another between the base figures A and B, the solver would expect an analogous move to take place between the target figure C and the correct answer figure. Or,

> Moves between non-transverse shape-arrangements correspond to moves in a category lattice built from non-accidental features in the arrangements, and analogies between arrangement moves proceed according to a codimension-preserving mapping between the respective lattices.

Category lattices of non-accidental features are described by Richards, Feldman, and Jepson [Richards et al., 1992]. In these lattices, a set of non-accidental features are enumerated for the shapes in question. If all of the features are completely independent, a complete boolean lattice can be formed from the binary presence or absence of each feature. However, if there are constraints among the features that preclude the appearance of some in the presence of others, then the lattice will not include every feature combination [Feldman, 1995].

The codimension of a level in the lattice is the number of "degrees of freedom" removed from the top-level category, which is the most arbitrary arrangement. If the locations of a set of shapes were parameterized, the codimension would be the number of parameters fixed by the presence of non-accidental features.

Each collection of shapes will have a different category lattice depending on the non-accidental features in their arrangements. For examples of lattices for pairs of shapes drawn from the set $\{point, line, circle, triangle\}$, see Table 2.

Arrangement transformations as lattice moves

A transformation applied to shapes that changes the perceptual category of their arrangement is a move in the corresponding category lattice. Such a move may occur from a category to one of higher or lower codimension, and it implies that the set of non-accidental features has changed.

The length of a lattice move is defined as the number of arcs on the shortest path from the source to the destination category in the lattice. Note that this distance may be different from the Hamming distance of the feature sets (the number of features added or removed from the initial set).

A shortest-length criterion provides a way of finding the best transformation between two arrangements, using the same argument that motivated choosing the identity transformation among similarity transformations: it is the simplest explanation (for some definition of simple) of how one arrangement came to be from the other.

Arrangement-based geometric analogies and lattice homomorphisms

Moves in category lattices describe the differences in two arrangements of shapes, in the same way that similarity transformations describe the differences in two shape poses. To form analogies between shape arrangements, a way is needed to relate a move in one feature lattice to another lattice, built of features in another set of shapes.

The relevant relationships are captured in a mapping from one lattice the other. A general lattice mapping is any function from the categories in one lattice to the categories in another lattice. Mappings useful for analogy are not arbitrary, but should preserve the structure of the lattices – the partial order of the categories dictated by their codimensions. The structure-preserving mapping is thus a homomorphism, or more formally:

Definition 1 Lattice homomorphism. *Let F and F' be sets of features $\{f_1,\ f_2,\ \ldots\ f_n\}$ and $\{f'_1,\ f'_2,\ \ldots\ f'_m\}$, and $C \subseteq 2^F$, $C' \subseteq 2^{F'}$ be category lattices defined on those features, possibly with formal constraints. Let $g : C^2 \rightarrow \{\text{true}, \text{false}\}$ and $g' : C'^2 \rightarrow \{\text{true}, \text{false}\}$ be lattice properties defined on pairs of categories in C and C', respectively. Then a lattice homomorphism $h : C \rightarrow C'$ is a function such that for every $c_1, c_2 \in C$,*

$$\textit{if } g(c_1, c_2) \textit{ then } g'(h(c_1), h(c_2)).$$

The natural structure to preserve between lattices is the relative ordering of categories by codimension (as in Figure 2). Using Definition 1, for the categories $c_1,\ c_2$ in C, $g(c_1, c_2)$ is true if and only if $codim(c_1) \leq codim(c_2)$, and similarly for g' defined over the categories in C'. Restating the general definition with this specific property,

Definition 2 Codimension-preserving lattice homomorphism. *Let C and C' be category lattices as defined above. Then a lattice homomorphism $h : C \rightarrow C'$ is codimension-preserving just in case for every $c_1,\ c_2$ in C*

$$\textit{if } codim(c_1) \leq codim(c_2) \textit{ in } C \textit{ then}$$
$$codim(h(c_1)) \leq codim(h(c_2)) \textit{ in } C'.$$

In geometric analogy problems, such as homomorphism allows three kinds of analogy among pairs of shape arrangements: pairs with lattice moves that increase codimension, moves that decrease codimension, and moves that keep it the same. The claim stated in the beginning of this section could thus be rephrased as the *codimension-change constraint*: a geometric analogy based on shape-arrangements preserves the direction of codimension change in the base and target category lattices.

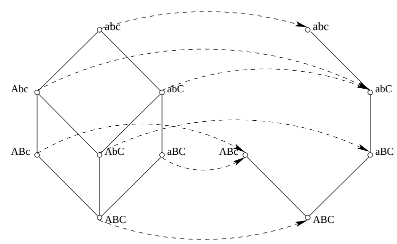

Figure 2: A codimension-preserving homomorphism from a full lattice to a constrained lattice, shown by the dotted lines.

This claim is illustrated in the first example, Table 1. At the bottom the (trivial) partial homomorphism from one lattice to the other indicates that 3 is the correct answer. The analogous lattice move in this case is one that keeps the codimension the same, which is not done in answer 2.

Note that a feature missing from the lattice for this example is the placement of the dot on the perimeter of the circle in alignment with the coordinate frame of the figure. These lattices could be augmented with such coordinate-frame aligned features, which would change the length (but not the direction) of the lattice moves.[1]

Examples

The set of examples shown in Table 3 is drawn from a restricted set of shapes $\{point, line, circle, triangle\}$. Each figure in the problem has exactly two of these shapes, and shapes are not added or removed from the figures. The category lattices for possible pairs of these shapes are shown in Table 2.

[1] This is another kind of "frame problem" – in this case, the determination of which features should be attended to in visual learning and analogy.

The shapes will have the same scale and orientation throughout, so that the correspondences and transformations in steps one through three of the solution process described above have already been solved. (A complete algorithm for this procedure would simultaneously search the space of similarity-transformation based correspondences and lattice moves.) The purpose of these examples is to illustrate the codimension-change constraint.

The examples, their corresponding lattice moves, and their codimension-preserving homomorphisms are shown in Table 3.

It is interesting to note that some of the solutions are ambiguous. For example, in problem 4, both solutions 2 and 3 correspond to a move to higher codimension in the triangle-circle lattice. In this case, the ambiguity is resolved in favor of the homomorphism that maps categories with the same concept, "touching." This suggests that favored solutions have analogous features as well as analogous lattice moves. Adding possible feature-to-feature correspondences to the model would further broaden the space of possible analogies in this domain.

Conclusion

This work claims that categorical arrangements of shapes in figures form a significant basis for geometric analogies. These analogies of non-transverse arrangements preserve the direction of codimension change in the corresponding category lattices of non-accidental features. It builds on the work of Richards and Feldman in defining those lattices and showing their relevance for perceptual categorization in observers [Richards et al., 1992].

It also extends Evans' work in developing a computational solution to geometric analogy problems by augmenting his solution to account for non-transverse shape arrangements in the problem figures. Although Evans' system computed coarse spatial relationships between shapes such as LEFT, RIGHT, ABOVE, BELOW, and INSIDE, the problems presented here require a richer model of shape arrangements. Although an implementation of the framework presented here is possible, automating the extraction of non-accidental features from arbitrary shape arrangements is a remaining research goal.

This work also studies visual analogical reasoning. Most work to date on analogical reasoning has assumed a fundamentally sentential representation of the models in question [Kedar-Cabelli, 1985]. However, the special characteristics of visual representations in human and computer problem-solving is a growing area of research [Glasgow et al., 1995]. Future questions along those lines that bear investigation include,

- *What is the relationship between sentential analogy and visual analogy for a particular domain?*
- *What advantages do visual analogies have for learning, recall, and problem-solving?*
- *Can the process of visual analogy be modeled algorithmically?*

Acknowledgements

I would like to thank Whitman Richards for comments on an earlier draft of this paper and for teaching the class which inspired this work, and Randy Davis for numerous improvements in exposition. This work was supported by the NDSEG Fellowship Program.

References

[Evans, 1968] Evans, T. G. (1968). Geometric analogy problems. In Minsky, M., editor, *Semantic Information Processing*. MIT Press.

[Falkenhainer et al., 1989] Falkenhainer, B., Forbus, K. D., and Gentner, D. (1989). The structure-mapping engine: Algorithm and examples. *Artificial Intelligence*, 41:1–63.

[Feldman, 1992] Feldman, J. (1992). *Perceptual Categories and World Regularities*. PhD thesis, Massachusetts Instituite of Technology.

[Feldman, 1995] Feldman, J. (1995). Formal constraints on cognitive interpretations of causal structure. In *IEEE Workshop on Architectures for Semiotic Modeling and Situation Analysis in Large Complex Systems*.

[Glasgow et al., 1995] Glasgow, J., Narayanan, N. H., and Chandrasekaran, B., editors (1995). *Diagrammatic Reasoning: Cognitive and Computational Perspectives*. The MIT Press.

[Johnson-Laird, 1983] Johnson-Laird, P. E. (1983). *Mental Models: Toward a New Science of Language, Inference, and Consciousness*. Harvard University Press.

[Kedar-Cabelli, 1985] Kedar-Cabelli, S. T. (1985). Analogy – from a unified perspective. Technical Report ML-TR-3, Rugters University Laboratory for Computer Science Research.

[Leishman, 1989] Leishman, D. (1989). Analogy as a constrained partial correspondence over conceptual graphs. In *Proceedings of the First Conference on Knowledge Representation and Reasoning*, pages 223–234.

[Richards et al., 1992] Richards, W., Feldman, J., and Jepson, A. (1992). From features to perceptual categories. In Hogg, D. and Boyle, R., editors, *British Machine Vision Conference*, pages 99–108. Springer-Verlag.

[Stiny, 1980] Stiny, G. (1980). Introduction to shape and shape grammars. *Environment and Planning B*, 7:343–351.

[Winston, 1980] Winston, P. H. (1980). Learning and reasoning by analogy. *Communications of the ACM*, 23(12).

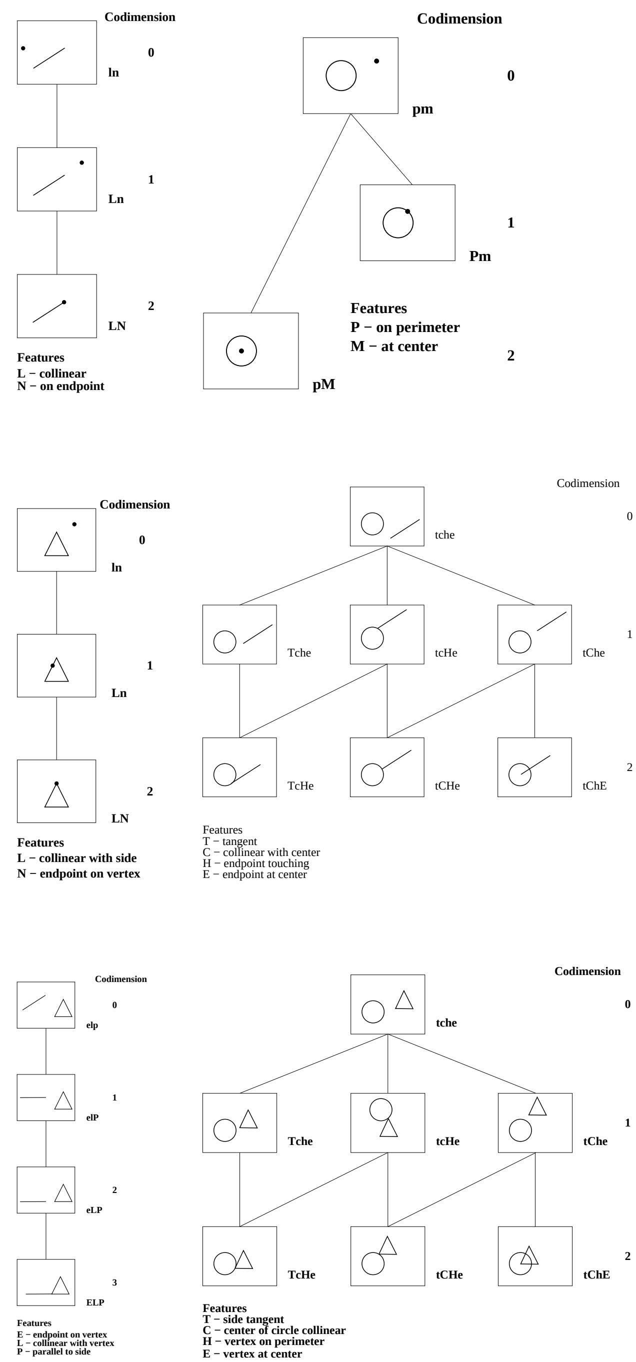

Table 2: Category lattices for pairwise non-accidental features. Upper case letters indicate the presence of a concept, and lower case letters its absence. The constraints for these incomplete lattices are omitted.

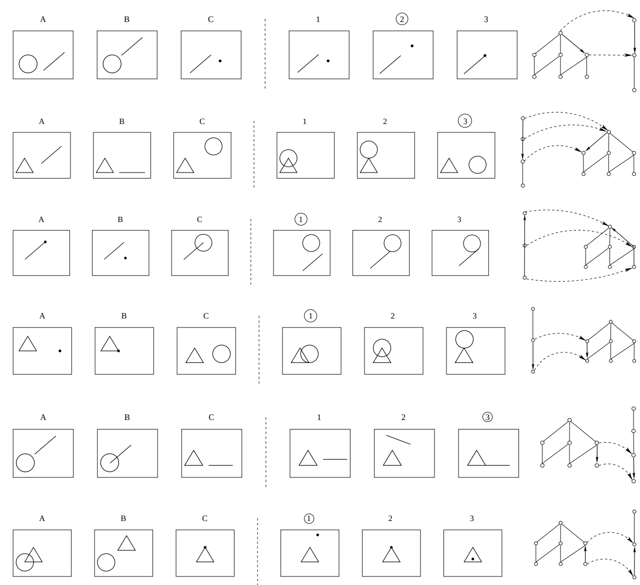

Table 3: Examples of analogies between non-transverse shape arrangements. The correct answer is circled for each problem. On the right of each problem, the left-hand lattice shows the lattice move from figure A to figure B. The right-hand lattice shows the correct lattice move from figure C to the answer figure. The dotted arrows show one possible codimension-preserving homomorphism from one lattice to the other.